ON THE INITIAL STAGE OF THE MOTION IN RAYLEIGH'S PROBLEM

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Rayleigh's problem is the problem of the flow of a gas near an infinite plate set impulsively in motion in its own plane with a velocity U. This problem is of considerable interest in connection with the study of the initial stage of arbitrary gas-dynamic motions, as well as in connection with many processes occuring in non-steady gas flows [1]. The theoretical results which are available are valid for the range $t/\Delta t \gg 1$ (Δt is the mean free time between molecular collisions and t is the time from the beginning of the motion), in which the momentum transfer in the bulk of the gas can be described in hydrodynamic terms [2]. But no less important is the initial stage of the motion, when $t/\Delta t \ll 1$ and only the collisions between the molecules and the plate are significant; this stage has been analyzed in considerable detail in [1]. According to [1], the collision integral in Boltzmann's equation can be neglected, i.e. it suffices to consider the equation

$$\frac{\partial f}{\partial t} + c_z \frac{\partial f}{\partial z} = 0, \tag{1}$$

where z is the coordinate normal to the plate.

Introducing the momentum exchange coefficient q, defined as the fraction of molecules diffusely reflected from the plate, and restricting our treatment, for simplicity, to the case of low M and equal gas and plate temperature, we can write the boundary condition for the distribution function in the form [1]

$$f(t, 0, c) = \frac{\rho}{(2\pi RT)^{3/2}} \left\{(1-q)\exp\left(-\frac{c^2}{2RT}\right) + q\exp\left[-\frac{(c_x-U)^2 + c_y^2 + c_z^2}{2RT}\right]\right\} \approx f_0(c)\left(1 + q\frac{c_x U}{RT}\right) \tag{2}$$

where ρ, T are the density and temperature of the gas, R is the gas constant, and $f_0(c)$ is the Maxwellian distribution. The initial condition is

$$f(0, z, c) = f_0(c). \tag{3}$$

Usually one assumes that U in (2) is a constant. This assumption, which is justified for the gas-dynamic range $t/\Delta t \gg 1$, is completely arbitrary for the initial stage $t/\Delta t \ll 1$. In fact, in a real process one can impose only the force which accelerates the plate. The speed of the plate is a function of time, and, in particular, the time required to accelerate the plate to the constant speed can be of the order of Δt. In that case the results of [1], based on the assumption that U is constant, lose their validity.

Therefore we assume that the force $\sigma(t)$ acting on unit area of the plate in the x direction is given, and the initial speed is

$$U(0) = 0. \tag{4}$$

To determine U(t) we use an equation which follows directly from Newton's second law. The momentum of the whole system per unit area of the plate is

$$P(t) = \gamma U(t) + 2\rho\int_0^\infty v(t, z)\,dz = \gamma U(t) + 2\int_0^\infty dz \iiint_{-\infty}^{\infty} f(t, z, c)\, c_x\, dc, \tag{5}$$

where v(t, z) is the mass velocity of the gas, and γ is the mass of the plate per unit area. The factor 2 in front of the integral is introduced to take into account the motion of the gas on both sides of the plate—for $z > 0$ and for $z < 0$. Clearly,

$$P(t) = \int_0^t \sigma(\tau)\,d\tau. \tag{6}$$

Equations (1) and (6), together with the conditions (2), (4), define the problem.

Taking the Laplace transform of (1) and taking into account (2) and (3), we obtain the transformed solution $f^\circ(p, z, c)$

$$f^\circ = f_0(c)\left[\frac{1}{p} + \frac{qc_x}{RT}U^\circ(p)\exp\left(-\frac{zp}{c_z}\right)y(c_z)\right],\qquad y(x) = \begin{cases}1, & x>0\\ 0, & x<0,\end{cases} \tag{7}$$

where y(x) is the Heaviside unit step function. The distribution function is then

$$f(t, z, c) = f_0(c) + \frac{qc_x}{RT}U\left(t - \frac{z}{c_z}\right)y\left(t - \frac{z}{c_z}\right)y(c_z). \tag{8}$$

The transformed equation (6) is

$$\gamma U^\circ(p) + 2\int_0^\infty dz \iiint_{-\infty}^{\infty} c_x f^\circ(p, z, c)\,dc = \frac{\sigma^\circ(p)}{p}.$$

Substituting f° from (7) into this equation and interchanging the order of integration with respect to z and c_z, we obtain

$$U^\circ(p)\left[\gamma + \frac{q\rho}{p}\left(\frac{2RT}{\pi}\right)^{1/2}\right] = \frac{\sigma^\circ(p)}{p}.$$

Assuming $\sigma(t) = \sigma$, $\sigma^\circ(p) = \sigma/p$, we obtain

$$U(t) = U_0\left[1 - \exp(-t/\tau)\right], \tag{9}$$

$$U_0 = \frac{\sigma}{q\rho}\left(\frac{\pi}{2RT}\right)^{1/2},\qquad \tau = \frac{\gamma}{q\rho}\left(\frac{\pi}{2RT}\right)^{1/2}. \tag{10}$$

Now we can easily calculate the moments which characterize the flow. The velocity, for example, is

$$v(t, z) = \frac{1}{\rho}\iiint_{-\infty}^{\infty} c_x f(t, z, c)\,dc = \frac{q}{2}U_0 \operatorname{erfc}\left(-\frac{\zeta}{t}\right) - \frac{q}{\sqrt{\pi}}U_0\exp\left(-\frac{t}{\tau}\right)\int_{\zeta/t}^{\infty}\exp\left(-u^2 + \frac{\zeta}{\tau u}\right)du. \tag{11}$$

In particular, the velocity of the gas at the plate is

$$v_0(t) = \frac{q}{2}U_0\left[1 - \exp\left(-\frac{t}{\tau}\right)\right],\qquad \zeta = \frac{z}{(2RT)^{1/2}}.$$

A simple calculation yields the shear stress

$$p_{xz} = q\rho U_0\sqrt{\frac{RT}{2\pi}}\left\{\exp\left[-\left(\frac{\zeta}{t}\right)^2\right] - 2\exp\left(-\frac{t}{\tau}\right)\int_{\zeta/t}^{\infty}u\exp\left(-u^2 + \frac{\zeta}{\tau u}\right)du\right\}. \tag{12}$$

The introduction of the finite acceleration of the plate from 0 to U yields the physically reasonable result that p_{xz} tends to 0 when $t/\tau \to 0$.

Analogous results hold for the other non-zero moments of $f(t, z, c)$—corrections to the static temperature of the gas, tangential and normal heat flux, etc. Analytic expressions for these variables can be easily obtained by integration of (8). These results reduce to the results for $U = U_0$ [1] for $t/\tau \to \infty$. The characteristic time of approach of these variables to their asymptotic values is τ.

The above results can easily be extended to the case when the temperatures of the gas and the plate are not equal. The difference between the two cases lies in the coefficients of the boundary condition (2), and leads to a minor change in the final expressions for U(t), $f(t, z, c)$, etc. For $t/\tau \to \infty$ these variables also reduce to those of [1].

The solution of Rayleigh's problem for $t/\Delta t \gg 1$ has contributed to the understanding of the behavior of the incompressible boundary layer far from the leading edge [1]. In an analogous way, the present solution corresponds to the boundary layer near the leading edge (in the region $x \ll \lambda/c_*$, where c_* is the mean thermal speed of the molecules). This range can become significant in the case of motion in a rarefied gas.

REFERENCES

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